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Validity of Integral Methods in MHD Boundary-Layer Analyses

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Several solutions of the MHD channel flow entrance problem[1–3] and the MHD flat-plate boundary layer[4–6] have been published recently which use the approximate momentum integral method. The attraction of the technique is the simplicity with which solutions to the nonlinear boundary-layer equations can be obtained. In view of the approximations involved in this approach, we feel that the results obtained require a more careful evaluation than they have often heretofore received before they can be accepted as being close to the exact solution to the problem. In this note we will review what we consider to be the minimum conditions under which such approximate solutions are acceptable and will ascertain the degree to which the solutions listed previously are adequate.

The reader is referred to Schlichting[7] for an excellent description of the method and for the criteria he uses to show that these approximate solutions for certain particular problems are acceptable. We can summarize these criteria as follows: 1) the approximate solution must agree well with experimental data and/or the exact, but more complicated analytical or numerical solution to the problem; 2) where no exact solution to the general problem exists, the approximate solution must agree with exact solutions for any limiting cases that may exist; and 3) the approximate solutions using several different, but physically reasonable velocity distributions (and temperature distributions for the thermal boundary layer) should show reasonable agreement with one another.

An additional requirement must be considered for MHD flow problems, since it is the differences between the behavior of the MHD flow and its ordinary hydrodynamic equivalent flow that are of the greatest interest. Thus, errors introduced by the approximations in the method should be small as compared with these differences in flow properties. Furthermore, care must be taken to base comparisons on the physically significant flow parameters; for example, for a channel flow these would be the displacement thickness δ^*, the pressure gradient dp/dx, and the friction factor f.

Laminar Momentum Boundary Layer

We now apply these criteria to the integral solution of the laminar MHD momentum boundary layer on an insulating wall.‡ For criterion 1 no experimental data are available, but finite difference solutions for MHD channel entrance flows have been obtained by Dix[8] for values of M^2/Re^2 of 10^{-2} and 10^{-3} (M is the Hartmann number), and by Shohet et al.[9] for $M = 4$. From the results of Ref. 9, values of δ^* and dp/dx can be obtained; Dix gives the wall shear stress. For criterion 2, two obvious asymptotic solutions exist. Close to the channel entrance the approximate results should be asymptotic to the Blasius solution or the ordinary hydrodynamic entrance flow solution since magnetic effects are small. At entrance lengths greater than the interaction length $\rho U/(\sigma B^2)$ the flow will become fully-developed Hartmann flow and exact analytical expressions for δ^*, dp/dx, and f can be obtained. All of these results should be used to evaluate the acceptability of the approximate solutions.

Although it is obvious from the asymptotic limits that the boundary-layer velocity profiles will not be similar along the length of the plate, two of the solutions listed assume similar velocity profiles. Maciulaitis and Loeffler[2] use a parabolic distribution that would be expected to be most appropriate for lengths that are small as compared to the interaction length. At the leading edge the friction factor for this parabolic solution differs from the exact Blasius solution by 12%. However the fully-developed solutions of Ref. 2 differ from the exact Hartmann solutions by 20%. From the data given in Ref. 2 the fully-developed value of δ/h (for $M = 10$) is 0.242, which gives $\delta^*/h = 0.081$ (compared to the Hartmann value of $1/M = 0.1$) and a fully-developed friction factor that is 0.81 times the exact Hartmann value. These differences are not apparent from this paper, because the authors compare only $u_c/U = (1 - \delta^*/h)^{-1}$ with the finite-difference solution of Shohet et al., thus obscuring the error in δ^*, and plot only the ratio of local friction factor to their own fully-developed friction factor (and not the exact Hartmann value) for comparison with the results of Dix. Moffatt[1] used the Hartmann profile for the velocity distribution in the developing boundary layer, an assumption that would be expected to give better agreement as the entrance length approaches the interaction length. His calculations of dp/dx for $M = 4$ are compared with those of Shohet et al. (difference less than 10%) and Roidt and Cess[10] (difference less than 5%) whose results are asymptotic to the Hartmann solution. At higher Hartmann numbers, the error close to the leading edge would be expected to increase.

To improve these results, the similar velocity profile assumption must be abandoned and profiles that change shape with x must be considered. Heywood[11] and Dhanak[3] have used the Pohlhausen profile (fourth-degree polynomial) in an attempt to follow the changing velocity distribution. The shape factor Λ for the MHD boundary layer is given by

$$\Lambda = \delta^2[\rho du_c/dx + \sigma B^2]/\mu \tag{1}$$

and Heywood has shown that, when $du_c/dx = 0$, Λ increases from zero at the leading edge to a value of 12 at $x' = 0.27$. (Here x' is length along the plate as a fraction of the interaction length, $x' = x\sigma B^2/(\rho U)$; thus the boundary layer should approach the Hartmann solution for values of x' greater than unity.) At this value of Λ the solution breaks down, since for $\Lambda > 12$ the velocity profile equation yields velocities inside the boundary layer greater than the freestream value. Dhanak's solutions are discontinued at the same point and, thus, never reach the Hartmann values. However, he continues his curves as a dashed line, but does not explain why his results are discontinued nor why an asymptotic solution is never reached. This Pohlhausen profile cannot adequately follow the change in velocity distribution required by the boundary-layer forces.

A more promising profile that has been suggested by the authors assumes that the local velocity distribution is the

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‡ The flow, field configuration and notation of Ref. 1 will be adopted in the present note.

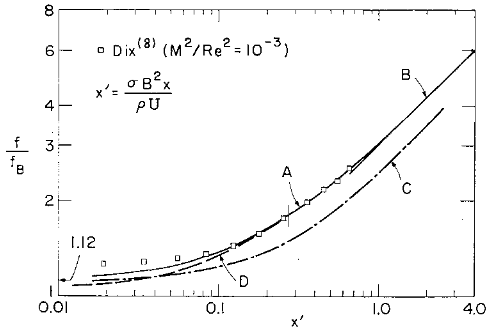

Fig. 1 Ratio of friction factor for MHD flow to the Blasius value for laminar boundary layer as a function of distance along plate.

Hartmann profile, in which the Hartmann number is based on the local boundary-layer thickness. It is given by

$$u/u_c = \{\cosh M' - \cosh[M'(1 - y/\delta)]\}/(\cosh M' - 1) \quad (2)$$

where $M' \equiv B\delta(\sigma/\mu)^{1/2}$. At the leading edge ($\delta \rightarrow 0$) this becomes a parabolic distribution; as $\delta \rightarrow h$ it becomes the fully-developed Hartmann flow. It has been used by Moffatt[4] for the case of $U = u_c =$ const.

All of these comments are illustrated in Fig. 1, where the local friction factor f for the MHD flat plate with u_c constant is divided by the Blasius solution $f_B = 0.664\ (\mu/\rho Ux)^{1/2}$ and plotted vs x'. Curve A was obtained using the velocity profile of Eq. (2). It is asymptotic to the Hartmann solution (line B) for $x' > 1$, and approaches a value of 1.12 as $x' \rightarrow 0$. Curve C was obtained with a parabolic velocity distribution and the error is 20% at $x' = 1$. Curve D shows the results of Heywood using the Pohlhausen profile. It approaches 1.03 as $x' \rightarrow 0$, but breaks down at $x' = 0.27$ although the results in the intermediate region are acceptable. Data from Dix's finite difference solution are also shown and the agreement with A is excellent (the channeling effect in Dix's solution is of order δ^*/h which is less than 0.03).

Turbulent Momentum Boundary Layer

The momentum integral method has also been used to analyze turbulent boundary-layer development, both on flat plates[4–6] and in channel entrances.[2] A summary of the experimental data of Hartmann and Lazarus, and Murgatroyd for fully-developed flow in rectangular channels is given by Harris,[12] and thus it is possible to apply criterion 1 to assess the accuracy of the analytical fully developed results.

Moffatt[5] assumed that the dependence of wall shear on boundary-layer thickness follows the Blasius turbulent friction law, and that the usual $\frac{1}{7}$-power velocity distribution may be used. The result is a simple closed-form solution for wall shear stress in terms of the ratio M/Re and x'. Kruger and Sonju[6] present a numerical solution based on boundary-layer velocity distributions semiempirically determined in Ref. 12. The authors compare their computed boundary-layer thickness with the results of Ref. 5, and show that discrepancies as large as a factor of 2 exist. However this comparison is unrealistic, as δ is not physically important and depends on the assumed velocity distribution. A comparison of the displacement thickness and wall shear stress in both the developing and fully developed regions[4] shows agreement to within 5%. This agreement satisfies criterion 3 and indicates that the results of Refs. 4–6 are acceptable.

Maciulaitis and Loeffler[2] have applied the physical model of Ref. 4 to the channel entrance problem. They take account of the change in centerline velocity and, thus, their fully developed solution can be compared with the experimental data, provided the asymptotic boundary-layer thickness is less than h. However, for channel flows in which the Hartmann number is sufficiently small so that the calculated value of δ grows to h, the predicted fully developed friction factor becomes the Blasius value, and MHD effects in the fully developed region are lost.

The results of Refs. 2 and 4–6 for fully developed friction factor are shown in Fig. 2 along with the experimental data of Ref. 12 for two values of the Reynolds number. A useful comparison of the results of Refs. 4–6 with the data can be made provided it is recognized that the flat-plate solutions are applicable only for values of M sufficiently large that the asymptotic value of δ^* is much less than h. From Ref. 4 it can be shown that this condition is satisfied when $32\ M/R_{DH} > 8/(R_{DH})^{5/8}$ where R_{DH} is the Reynolds number based on the hydraulic diameter. The vertical lines in Fig. 2 show the lower bounds on $32\ M/R_{DH}$ for the two values of R_{DH} considered. For turbulent flow it is also required that $32\ M/R_{DH} < 0.032$. For values of M and R_{DH} that satisfy these restrictions, the results of Refs. 4–6 for the fully developed region agree with the experimental friction factors to within 25%. The channel flow results of Ref. 2 agree to within 8% of the experimental values, where again Maciulaitis and Loeffler have discontinued their curves when $\delta = h$.

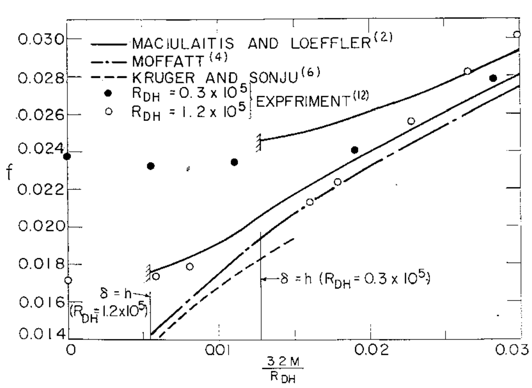

Fig. 2 Fully developed friction factor for MHD turbulent boundary layer as a function of $32M/R_{DH}$.

Laminar Thermal Boundary Layer

To the authors' knowledge, integral solutions for the laminar MHD heat-transfer problem have been attempted in only four instances.[3–5,11] In view of the complete lack of experimental data and the multiplicity of parameters required to describe the flow, application of the forementioned criteria is difficult. Romig[13] gives asymptotic solutions for the local heat-transfer rate at constant wall temperature in the case of fully developed Hartmann flow. However, since the solutions presented in Refs. 4, 5, and 11 show results for integrated heat fluxes only, and since they include the effects on streamwise temperature variation of the work done by the fluid against Lorentz forces, the results are not strictly comparable with those of Ref. 13. Furthermore, it was shown in Ref. 11, where the energy integral equation was solved by using both parabolic and Pohlhausen-like temperature profiles, that these assumed distributions yield results that differ by the same order as the difference between MHD and ordinary hydrodynamic heat-transfer rates. Clearly, criterion 3 is not satisfied, and until additional efforts are made to develop physically realistic temperature profiles within the boundary layer, the present results must be treated as being purely conjectural. In this context, note that the momentum and energy equations for the MHD boundary layer are not of similar form (as they are for the ordinary flat-plate flow with viscous dissipation neglected), and therefore the velocity and temperature profiles will not be similar. Furthermore, at the wall, the energy equation becomes

$$\lambda(\partial^2T/\partial y^2) + \mu(\partial u/\partial y)^2 + \sigma u_c^2B^2K^2 = 0 \quad (3)$$

and temperature profiles will only be similar along the plate if viscous dissipation is neglected and the electrodes are short-circuited ($K = 0$).

Dhanak[3] has attempted to evaluate the heat transfer in the entrance region of an MHD channel using a fourth-degree polynomial for the boundary-layer temperature profile, and concludes that ohmic dissipation is not an important consideration in the cases he examined. However his calculations are terminated at values of $x' = 0.27$ and 0.10 for Hartmann numbers of 30 and 10, respectively. Since, magnetic effects do not play the dominant role until $x' \sim 1$, Dhanak's results do not indicate the possible importance of the MHD effects. He also considers only the case where the electrodes are short-circuited, and Refs. 4, 5, and 11 have shown that for $K = 0$ the increase in heat-transfer coefficient above the ordinary hydrodynamic flow is less than the corresponding increase for $K > 0$.

Conclusions

We emphasize that we have attempted to show how the results obtained with the integral method should be evaluated and not that the technique itself is not applicable to MHD flows. Figure 1 demonstrates that acceptable integral solutions for the laminar momentum boundary layer now exist. Figure 2 indicates that for a limited range of M and Re, the fully developed solutions for the turbulent boundary layer are satisfactory, and thus the results in the developing region may be tentatively accepted. However for the laminar thermal boundary layer no satisfactory solution exists, and the need for an evaluation of various temperature profiles with the velocity profile of Eq. (2) is obvious. Finally, experimental data for the developing boundary layer would provide the best test of the integral method results.

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Erratum: "Ablation Mechanisms in Plastics with Inorganic Reinforcement"

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Erratum: "Theory for the Ablation of Fiberglas-Reinforced Phenolic Resin"

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IN these papers kinetic data on the rate of reaction of carbon and silica at high temperature were used as part of the input information for calculation of the rate of ablation under the proposed mechanism.

Recently, the work of Blumenthal et al.,[1] in the study of these high-temperature carbon-silica reactions, has come to our attention. His work indicated a reaction rate about four orders of magnitude lower than those obtained in the preliminary experiments reported by us.

As a result of this apparent discrepancy, we have reviewed our data very carefully and find that a serious error was made in interpretation which produced an error of approximately four orders of magnitude in the rate constant. On page 538 of the article published in the ARS Journal, the rate constant k is reported as 2×10^{14} g/cm^3-min. As corrected it should be approximately 1.0×10^{10} g/cm^3-min.

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